

Entangled multi-qubit states without higher-tangle

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We analyze mixed multi-qubit states composed of a W class state and a product state with all qubit in $|0\rangle$. We find the optimal pure state decomposition and convex roofs for higher-tangle with bipartite partition between one qubit and the rest qubits for those mixed states. The optimality of the decomposition is ensured by the Coffman-Kundu-Wootters (CKW) inequality which describes the monogamy of quantum entanglement. The generalized monogamy inequality is found to be true for W class states with arbitrary partitions between one qubit and multi-qubit.

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Quantum entanglement has been the subject of much study in recent years as a physical resource for quantum communication and quantum information processing [1]. Entangled states have a number of remarkable features which has inspired an enormous literature in the years since their discovery. As a consequence, the study of quantum entanglement from various view points has been a very active area and has led to many interesting results. However, only the pure state entanglement shared between two parties is thoroughly understood and quantified; progress on mixed state of higher-dimension and the multipartite state has been much slower. In this paper, we will study the higher-tangle of mixed multi-qubit states and the related monogamy of entanglement.

Monogamy of entanglement is a key property discovered recently in the context of multiqubit entanglement [2, 3]. It states that unlike classical correlations, quantum entanglement cannot be freely shared among many parties. For example, if a pair of qubits Alice (A) and Bob (B) have perfect quantum correlation, namely, if they are maximally entangled, then Alice cannot be entangled to a third party Charlie (C). The monogamy inequality is to capture in a quantitative way the trade-off between quantum entanglement shared by pair (A, B) and by pair (A, C). In the context of quantum cryptography [4], such a monogamy property is of fundamental importance since it quantifies how much information an eavesdropper could potentially obtain about the secret key to be extracted. The triqubit monogamy inequality was first proposed and proved by Coffman, Kundu and Wootters (CKW) in their seminal paper [2], and it is also named as CKW inequality. Recently the long standing conjecture of the general monogamy inequality with bipartite partition between one qubit and the rest qubits for multiqubit states is proved by Osborne and Verstraete [5]. To be precise, a multiqubit state $\rho_{AB_1\dots B_n}$ shared among $n+1$ parties, the distribution of bipartite entanglement satisfies the monogamy inequality (also refer as CKW inequality):

$$\tau(\rho_{AB_1}) + \tau(\rho_{AB_2}) + \dots + \tau(\rho_{AB_n}) \leq \tau(\rho_{A:B_1B_2\dots B_n}), (1)$$

where the bipartite quantum entanglement is measured by *tangle* τ which is the square of the well known *concurrence* [6, 7]. For $\tau(\rho_{A:B_1B_2\dots B_n})$, the entanglement with bipartite partition for multiqubit is across $A : B_1B_2\dots B_n$ cut. It is obvious from this monogamy inequality that the summation of quantum entanglement measured by tangle in pairwise type $\tau(\rho_{AB_1}), \dots$, and $\tau(\rho_{AB_n})$ is upper bounded by the amount of entanglement with bipartite partition between A and $B_1\dots B_n$ measured by $\tau(\rho_{A:B_1B_2\dots B_n})$.

In the monogamy inequality (1), the bipartite entanglement is measured by tangle τ which is the square of the concurrence for pure states, as we mentioned. The *concurrence* introduced in Refs.[6, 7] is directly related with entanglement of formation [8] for two-qubit case. As we know, the entanglement of formation is a well accepted entanglement measure, and has a clear physical implication in quantum communication protocols such as quantum teleportation [9]. However, in general, the *concurrence*, also with entanglement of formation can only be explicitly calculated for pure states and two-qubit mixed states. The reason is that the *concurrence* and the entanglement of formation of a mixed state are represented by the convex roof of the pure state decompositions [10]. As we know, there is no general method to find the optimal pure state decomposition for a mixed state. However, recently, Lohmayer *et al* [11] successfully find the optimal decomposition of mixed three-qubit states composed of a GHZ state and a W state. And the optimal decomposition and the convex roofs for the three-tangle τ_3 [2] are obtained, where three-tangle for three-qubit state ρ_{ABC} is defined as $\tau_3(\rho_{ABC}) \equiv \tau(\rho_{A:BC}) - \tau(\rho_{AB}) - \tau(\rho_{AC})$, the residual entanglement between A and BC that cannot be accounted for by the entanglement of A with B and C separately. Similarly, we refer $\tau(\rho_{A:B_1\dots B_n}) - \sum_{j=1}^n \tau(\rho_{AB_j})$ to higher-tangle. The monogamy inequalities by other entanglement measures and related topics can be found in Refs.[12, 13, 14, 15, 16, 17]

In this paper, we shall start from the analyzing of the tangle for $(n+1)$ -qubit *mixed* states composed of a W

class state and the state $|0\rangle^{\otimes(n+1)}$. The optimal decomposition for this kind of mixed states is found and the optimality is ensured by the CKW inequality. We then consider a more general monogamy inequality for a W class state with arbitrary partitions which is beyond the scope of the case proved in Ref.[5]. Really, by the obtained result of tangle of the mixed state under consideration, the monogamy inequality is found to be still true for this case. And the monogamy inequalities by other entanglement measures will be presented. The study of this topic may shed light into both the entanglement measure for multipartite mixed states, a well known problem in entanglement theory of quantum information processing, and the monogamy in quantum entanglement distribution. More references about those problems can be found in a nice recent review paper of quantum entanglement in Ref.[1].

Tangle of multiqubit mixed states.—Let's consider a $(n+1)$ -qubit W class state defined as:

$$|W\rangle = a|100\dots0\rangle + b_1|010\dots0\rangle + b_2|001\dots0\rangle + \dots + b_n|000\dots1\rangle, \quad (2)$$

where generally $a, b_j, j = 1, \dots, n$ are complex numbers. Without loss of generality, we assume that they are real numbers. This assumption does not change any of our conclusions in this paper. And as usual, we have the normalization condition $a^2 + \sum_{j=1}^n b_j^2 = 1$. Suppose we have a $(n+1)$ -qubit mixed state shared by parties A and B_1, B_2, \dots , and B_n which is a mixture of the state $|W\rangle$ and the state $|\vec{0}\rangle \equiv |0\rangle^{\otimes(n+1)}$,

$$\rho_{AB_1\dots B_n} = p|W\rangle\langle W| + (1-p)|\vec{0}\rangle\langle\vec{0}|, \quad (3)$$

where p is the probability of the W class state. We would like to find the bipartite state tangle $\tau(\rho_{A:B_1B_2\dots B_n})$ between A and $B_1B_2\dots B_n$.

Before proceed, let's introduce the definition of the concurrence. For a pure bipartite state $|\phi_{AB}\rangle$, the concurrence is defined as $\mathcal{C}(\phi_{AB}) = 2\sqrt{\det\rho_A}$, where $\det\rho_A$ is the determinant of matrix ρ_A , $\rho_A = \text{tr}_B(\phi_{AB})$ is the reduced density operator of ϕ_{AB} , and tr_B is to take trace over Hilbert space B . Here we use the notation $\phi_{AB} = |\phi_{AB}\rangle\langle\phi_{AB}|$. The concurrence of the mixed state ρ_{AB} is defined as the average pure state concurrence minimized over all pure state decompositions,

$$\mathcal{C}(\rho_{AB}) = \min \sum p_j \mathcal{C}(\phi_{AB}^j), \quad (4)$$

which is also called convex roof extension [10], where $\rho_{AB} = \sum p_j \phi_{AB}^j$. The optimal decomposition means that the pure state decomposition which gives the defined mixed state concurrence. The optimal decomposition is not necessarily unique. For two-qubit mixed state, the concurrence can be found by a well known method introduced by Wootters [7]. However, for the general mixed state of multi-qubit, there does not exist an explicit formula for the concurrence.

Tangle of a pure state is defined as the squared concurrence. Similarly tangle of a mixed state is defined as the average pure state tangle over all pure state decompositions,

$$\tau(\rho_{AB}) = \min \sum p_j \tau(\phi_{AB}^j) = \min \sum p_j \mathcal{C}^2(\phi_{AB}^j). \quad (5)$$

We next will study the tangle of the mixed state in (3), $\tau(\rho_{A:B_1\dots B_n})$. We remark that though $\rho_{AB_1\dots B_n}$ is a multiqubit state, the tangle $\tau(\rho_{A:B_1\dots B_n})$ is for a bipartite partition between A and $B_1\dots B_n$.

Stimulated by the work in Ref.[11], see [26], we propose a trial pure state decomposition for the mixed state in (3) as the following:

$$\rho_{AB_1\dots B_n} = \frac{r}{3} \sum_{k=0}^2 |\psi^k\rangle\langle\psi^k| + (1-r)|\vec{0}\rangle\langle\vec{0}|, \quad (6)$$

where $|\psi^k\rangle = \sqrt{q}|W\rangle + \sqrt{1-q}\omega^k|\vec{0}\rangle$, and $\omega = e^{\frac{2\pi i k}{3}}$. Compare this pure state decomposition with the state in (3), we find that equation $rq = p$ should be satisfied. For pure state $|\psi^k\rangle$, the reduced density operator of qubit A , $\rho_A^k = \text{tr}_{B_1\dots B_n} \psi^k$, takes the form

$$\rho_A^k = (\sqrt{q}a|1\rangle + \sqrt{1-q}\omega^k|0\rangle)(\sqrt{q}a|1\rangle + \sqrt{1-q}\omega^k|0\rangle)^\dagger + q \sum_j b_j^2 |0\rangle\langle 0|. \quad (7)$$

By the equation $\tau(\psi^k) = 4\det\rho_A^k$, the tangle of the pure state $|\psi^k\rangle$ with bipartite partition across $A : B_1\dots B_n$ cut can be calculated as $\tau(\psi^k) = 4q^2a^2(\sum_j b_j^2)$.

By definition we know that tangle $\tau(\rho_{A:B_1\dots B_n})$ should be less than or equal to the average tangle in the trial pure state decomposition (6),

$$\begin{aligned} \tau(\rho_{A:B_1\dots B_n}) &\leq \frac{r}{3} \times \sum_{k=0}^2 \tau(\psi^k) \\ &= 4rq^2a^2 \left(\sum_j b_j^2 \right) \\ &= 4pqa^2 \left(\sum_j b_j^2 \right), \end{aligned} \quad (8)$$

where the last equation is due to the condition $rq = p$, and also when $r = 1$, q is minimum $q = p$, we thus know $\tau(\rho_{A:B_1\dots B_n}) \leq 4p^2a^2(\sum_j b_j^2)$. We next will show that this is a tight upper bound for tangle, e.g., the trial pure state decompositions can realize the optimal pure state decomposition.

According to the CKW inequality (1) proved in Ref.[5], we know that $\tau(\rho_{A:B_1\dots B_n}) \geq \sum_{j=1}^n \tau(\rho_{AB_j}) \geq \sum_{j=1}^n \mathcal{C}^2(\rho_{AB_j})$, where the last inequality is because that the squared concurrence is a convex function on the set of density matrices [2], it is further shown to be an inequality in Ref.[18]. With the help of Eq.(3), we obtain

$$\rho_{AB_j} = p(a|10\rangle + b_j|01\rangle)(a|10\rangle + b_j|01\rangle)^\dagger$$

$$+[1-p+\sum_{k \neq j} b_k^2]|00\rangle\langle 00|. \quad (9)$$

The squared concurrence can be calculated as $\mathcal{C}^2(\rho_{AB_j}) = 4a^2b_j^2p^2$ by the formula in Ref.[7]. So we find the lower bound of the tangle has the form $\tau(\rho_{A:B_1...B_n}) \geq 4p^2a^2(\sum_j b_j^2)$. Combine the upper bound and lower bound, we have a conclusion that

$$\begin{aligned} \tau(\rho_{A:B_1...B_n}) &= 4p^2a^2(\sum_j b_j^2) \\ &= \sum_{j=1}^n \tau(\rho_{AB_j}). \end{aligned} \quad (10)$$

We thus know that the optimal pure state decomposition of tangle for state $\rho_{A:B_1...B_n}$ can be in the form presented in Eq.(6). Actually only three vectors are enough to realize the optimal pure state decomposition since $r = 1$ is for all region of p , where $p \in [0, 1]$. Since the monogamy inequality (1) is tight for the case considered in this paper as shown in (10), the higher-tangle defined as $\tau(\rho_{A:B_1...B_n}) - \sum_{j=1}^n \tau(\rho_{AB_j})$ vanishes for the state in (3), while this mixed state is in general entangled except in some extreme cases.

Monogamy inequality with arbitrary partitions for W class states.—A nature question arise concerning about the CKW inequality in (1) is whether this monogamy inequality is generally true or not for higher-dimensional systems. A simple example in Ref.[15] shows that this monogamy inequality does not hold in general for higher-level quantum states. Still we may wonder, to what extent, this monogamy inequality can be applied, for example, even for case of multiqubit states. We next consider a following question: A multiqubit state $\rho_{ABCD...}$ shared by A, B, C, D, \dots , etc., while particle in A is a qubit, but B, C, D, \dots , contains several qubits as $B = (B_1B_2...B_n)$, $C = (C_1C_2...C_m)$, $D = (D_1D_2...D_l), \dots$. From the CKW inequality we know that both $\tau(\rho_{A:BCD...})$ and $\tau(\rho_{A:B}) + \tau(\rho_{A:C}) + \tau(\rho_{A:D}) + \dots$, are greater than or equal to quantity $\sum_{j_B} \tau(\rho_{AB_{j_B}}) + \sum_{j_C} \tau(\rho_{AC_{j_C}}) + \sum_{j_D} \tau(\rho_{AD_{j_D}}) + \dots$. The problem is for arbitrary partitions, whether we still have the following monogamy inequality?

$$\tau(\rho_{A:BCD...}) \geq \tau(\rho_{A:B}) + \tau(\rho_{A:C}) + \tau(\rho_{A:D}) + \dots \quad (11)$$

We may consider a normalized W class state shared by $ABCD...$ as the follows,

$$\begin{aligned} |\tilde{W}\rangle &= \tilde{a}|100...0\rangle + [\sum_{j_B} \tilde{b}_{j_B} \sigma_{j_B}^x + \sum_{j_C} \tilde{c}_{j_C} \sigma_{j_C}^x \\ &\quad + \sum_{j_D} \tilde{d}_{j_D} \sigma_{j_D}^x + \dots]|\vec{0}\rangle, \end{aligned} \quad (12)$$

where σ_k^x is the Pauli matrix on site k . This state is actually just like the state in (2). By direct calculation and as pointed out in [2], we know the W class states saturate the CKW inequality. That means $\tau(\rho_{A:BCD...}) =$

$$\begin{aligned} \tau(A:BCD...) &= \\ \text{○} : \text{○ ○ ○ ○ ○ ○ ○ ○ ○ ○} \dots & \\ A \quad B \quad C \quad D & \\ \tau(A:B) + \tau(A:C) + \tau(A:D) + \dots & \\ \text{○} : \text{○ ○ ○ ○} + \text{○} : \text{○ ○ ○} + \text{○} : \text{○ ○ ○ ○} + \dots & \\ A \quad B \quad A \quad C \quad A \quad D & \end{aligned}$$

FIG. 1: A generalized CKW inequality is satisfied (saturated) for mixed states in (3) form, if $p = 1$, they reduce to W class states.

$\sum_{j_B} \tau(\rho_{AB_{j_B}}) + \sum_{j_C} \tau(\rho_{AC_{j_C}}) + \sum_{j_D} \tau(\rho_{AD_{j_D}}) + \dots$ for state $|\tilde{W}\rangle$ in Eq. (12). Then it is quite interesting to know whether all of the states $\rho_{AB}, \rho_{AC}, \rho_{AD}, \dots$, saturate the CKW inequality or not since we already know $\tau(\rho_{A:BCD...}) \leq \tau(\rho_{A:B}) + \tau(\rho_{A:C}) + \tau(\rho_{A:D}) + \dots$. The saturation means that this is an equation, otherwise, the CKW inequality for partition $\rho_{A:BCD}, \rho_{A:B}, \rho_{A:C}, \dots$ will be violated. Thus W class states in (12) are good candidates to check the CKW inequality since there is no room for other than saturation case, otherwise, a counterexample is found to violate the CKW inequality.

Without lose of generality, let's study the reduced multiqubit mixed state ρ_{AB} , from Eq.(12), we find

$$\begin{aligned} \rho_{AB} &= (\tilde{a}|10...0\rangle + \tilde{b}_1|01...0\rangle + \dots + \tilde{b}_n|00...1\rangle) \times \\ &\quad (\tilde{a}\langle 10...0| + \tilde{b}_1\langle 01...0| + \dots + \tilde{b}_n\langle 00...1|)^\dagger \\ &\quad + (\sum_{j_C} \tilde{c}_{j_C}^2 + \sum_{j_D} \tilde{d}_{j_D}^2 + \dots)|0...0\rangle\langle 0...0|. \end{aligned} \quad (13)$$

Let $p = \tilde{a}^2 + \sum_j \tilde{b}_j^2$, and let $a = \tilde{a}/\sqrt{p}$, $b_j = \tilde{b}_j/\sqrt{p}$, quite interestingly, we find $\rho_{AB} = \rho_{AB_1...B_n}$, as presented in (3). Thus, all mixed states $\rho_{AB}, \rho_{AC}, \rho_{AD}, \dots$, are exactly the same form as the mixed states in (3).

We already find the optimal pure state decomposition of the multiqubit mixed state $\rho_{AB_1...B_n}$, and really it saturates the CKW inequality, e.g., without higher-tangle. We thus know that all mixed states $\rho_{AB}, \rho_{AC}, \rho_{AD}, \dots$, saturate the CKW inequality. So for W class states, the generalized CKW inequality (11) still holds for bipartite case with one party to be a qubit and other parties can be in arbitrary partitions B, C, D, \dots . Interestingly, the CKW inequality is saturated for all of those partitions. One may realize now that not only W class states, actually all states (mixed) in from (3) actually saturate the CKW inequality with arbitrary partitions, see Figure. Here, we are tempted to conjecture that for all multiqubit systems, a more general monogamy inequality as (11) holds. This question of course should be explored further.

An example.—Let's consider a five-qubit W state, $|W_{ABC}\rangle = (|10000\rangle + |01000\rangle + \dots + |00001\rangle)/\sqrt{5}$, where $B = B_1B_2$ and $C = C_1C_2$. Since W state is symmetric, and we have $\rho_{AB} = \rho_{AC}$. Our motivation is

to find whether we have a general monogamy inequality as $\tau(\rho_{A:BC}) \geq \tau(\rho_{A:B_1B_2}) + \tau(\rho_{A:C_1C_2})$. One can find $\tau(\rho_{A:BC}) = 4\det\rho_A = \frac{16}{25}$. Since W state saturates the CKW inequality, then if $\tau(\rho_{A:B}) \neq \frac{8}{25}$, see [27], it is a counter-example that violates the general CKW inequality. For W state, we have $\rho_{AB_1B_2} = \frac{3}{5}|W'\rangle\langle W'| + \frac{2}{5}|000\rangle\langle 000|$, where we denote $|W'\rangle \equiv (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$. With our conclusions in this paper, the optimal pure state decomposition takes the form $\rho_{A:B_1B_2} = \frac{1}{3}\sum_{k=0}^2 |\psi'^k\rangle\langle\psi'^k|$, where $|\psi'^k\rangle = \sqrt{\frac{3}{5}}|W'\rangle + \sqrt{\frac{2}{5}}\omega^k|000\rangle$. Then by definition in (5), we have $\tau(\rho_{A:B_1B_2}) = \frac{8}{25}$. Thus for W state, a more general monogamy inequality is satisfied.

Monogamy inequality of entanglement by other entanglement measures.— Three-tangle τ_3 is defined due to the CKW inequality [2, 12] as the residual entanglement. Since it vanishes for any separable pure state, it is then proposed to define the genuine multi-qubit entanglement [17]. However, W states (W class states) which are genuine multipartite entangled have vanishing three-tangle, as we already know. We thus propose to use other monogamy inequalities in terms of different entanglement measures to quantify the residual entanglement. In Ref.[16], the monogamy inequality by entanglement measure, negativity denoted as $\mathcal{N}(\rho_{AB})$ [19], is proposed. It is shown that the residual entanglement is always larger than zero for W like states. Similar as in Ref.[16], we would like to point out that, the entanglement measure by the realignment method [20], see also [21], denoted as $\mathcal{R}(\rho_{AB})$, also satisfies the monogamy inequality. To be precise, let $|\psi_{ABC}\rangle$ be a triqubit state, then we have

$$\mathcal{R}^2(\psi_{A:BC}) \geq \mathcal{R}^2(\rho_{AB}) + \mathcal{R}^2(\rho_{AC}), \quad (14)$$

similar as the monogamy inequality in terms of the negativity $\mathcal{N}^2(\psi_{A:BC}) \geq \mathcal{N}^2(\rho_{AB}) + \mathcal{N}^2(\rho_{AC})$. Those two inequalities are due to the fact that, $\mathcal{R}(\rho_{AB})$ and $\mathcal{N}(\rho_{AB})$ are lower bounds for concurrence $\mathcal{C}(\rho_{AB})$ as pointed out in Ref.[22], $\mathcal{C} \geq \max\{\mathcal{N}, \mathcal{R}\}$. Thus the proof of CKW inequality leads to the proof of those two inequalities. Those two inequalities can be complementary to each other. We remark that one advantages to use $\mathcal{N}, \mathcal{R}(\rho_{AB})$ is that they are operational to be calculated. The monogamy inequality by relative entropy of entanglement [23, 24] is also an interesting question [25].

Conclusions.—We analyze the multiqubit mixed states composed of a W class state and the state $|\vec{0}\rangle$, the optimal pure state decomposition is found for tangle for those states. It is shown that these mixed states saturate the CKW inequality and thus are without higher-tangle. We then study a more general CKW inequality for multiqubit state with arbitrary partitions, W class states are likely to violate this general CKW inequality. However, we find the general CKW inequality with arbitrary partitions is still true for W class states, the general CKW inequality

is saturated by W class states. A new monogamy inequality by matrix realignment quantity is presented.

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- [1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Quantum entanglement*, quant-ph/0702225, submitted to Rev. Mod. Phys.
 - [2] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).
 - [3] B. M. Terhal, IBM J. Res. Dev. **48**, 71 (2004),
 - [4] A. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
 - [5] T.J.Osborne,F.Verstraete, Phys.Rev.Lett.**96**,220513(2006).
 - [6] S.Hill and W.K.Wootters, Phys.Rev.Lett.**78**, 5022(1997).
 - [7] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
 - [8] C. H. Bennett, D. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
 - [9] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
 - [10] A. Uhlmann, Phys. Rev. A **62**, 032307 (2000).
 - [11] R. Lohmayer, A. Osterloh, J. Siewert, and A. Uhlmann, Phys. Rev. Lett. **97**, 260502 (2006).
 - [12] A. Miyake, Phys. Rev. A **67**, 012108 (2003).
 - [13] M.Koashi,A.Winter,Phys.Rev.A**69**, 022309(2004).
 - [14] M.Christandl and A.Winter, J.Math.Phys.**45**, 829(2004).
 - [15] Y. C. Ou, Phys. Rev. A **75**, 034305 (2007).
 - [16] Y. C. Ou and H. Fan, Phys. Rev. A **75**, 062308 (2007).
 - [17] W. Dür, G.Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
 - [18] T.J.Osborne, Phys.Rev.A**72**,022309(2005).
 - [19] G.Vidal and R.F.Werner,Phys.Rev.A**65**, 032314(2002).
 - [20] K. Chen, and L. A. Wu, QIC **3**, 193 (2003).
 - [21] H. Fan, e-print arXiv:quant-ph/0210168.
 - [22] K. Chen, S. Albeverio, and S. M. Fei, Phys. Rev. Lett. **95**, 040504 (2005).
 - [23] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).
 - [24] V. Vedral, Rev. Mod. Phys. **74**, 197 (2002).
 - [25] H. Fan, and M. B. Plenio, in preparation.
 - [26] In Ref.[11], the authors study the triqubit mixed state composed of W state $(|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$ and the GHZ state $(|000\rangle + |111\rangle)/\sqrt{2}$. This case is different from our case in several aspects: Our case is a multiqubit state; We use the W class state which has different amplitude in each term compared with the W state; We use product state of $|0\rangle$ instead of the GHZ state in the mixture; The final conclusion is different
 - [27] Apparently it should be strictly larger than $8/25$ since the CKW inequality in (1), with this we have $\tau(\rho_{A:B_1B_2}) \geq 2\tau(\rho_{AB_1}) = \frac{8}{25}$.